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A NOTE ON THE OVERFLOW PROCESS FROM A FINITE MARKOVIAN QUEUE

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A note on the overflow process from a finite Markovian queue *)

by

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ABSTRACT

In this note we determine the distribution of the time between overflows for a single server queueing system with finite waiting room and state-dependent service and arrival rates. As an application we analyze a GI/M/∞ system where the arrival process is the overflow process from the M/M/s/r queue.

KEY WORDS & PHRASES: *overflow process, Markovian queue, infinite-server queue*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

Consider a GI/G/s/r queueing system (s servers, r waiting places) where $0 < s < \infty$ and $0 \leq r < \infty$. If a customer arrives to find s + r customers in the system he departs never to return, and he is then said to have overflowed. Otherwise he enters the system and, depending on whether there are free servers or not, is served immediately or occupies a free waiting place until his turn to be served comes up, the order in which waiting customers are served being FCFS, say. Our interest centers on the point process of overflowing customers which will be denoted by $(GI/G/s/r)_{overflow}$ and called the overflow process from the GI/G/s/r queue.

The study of overflow processes is of importance in teletraffic theory, since telephone systems usually provide for alternative routes for calls that are blocked on a specific trunk group. In this context PALM [32] studies the system GI/M/1/0 and shows that the overflow process is a renewal process. Further, he relates the Laplace-Stieltjes transform of the interoverflow time distribution to that of the interarrival time distribution. Since the overflow process from a GI/M/s/0 system, where s > 1, may be conceived as the overflow process from a $(GI/M/s-1/0)_{overflow}/M/1/0$ system, Palm's analysis actually pertains to GI/M/s/0 for all s > 0. We refer to KHINTCHINE [19], TAKÁCS [41, 42], BENEŠ [3], RIORDAN [37], PEARCE and POTTER [33], WALLIN [43] and POTTER [34] for treatments of Palm's theory and its ramifications. Several of these authors, including Palm, give detailed results for the overflow process from the system M/M/s/0 (see DESCLOUX [10] for related results).

Adding a finite number r of waiting places to the GI/M/s/0 system complicates the analysis considerably, for, although the overflow process is still renewal, an iterative argument as when r = 0 is no longer valid. The case $s = 1$, $0 \leq r < \infty$ was treated by ÇINLAR and DISNEY [5], while for arbitrary s and r only recently McNICKLE [26] and DE SMIT [39] have derived an explicit expression for the Laplace-Stieltjes transform of the interoverflow time distribution.

An even more complicated situation arises when one assumes non-exponential service time distributions, since then the overflow process is not in general a renewal process. The only available results are those of HALFIN [12] who studies the overflow process from a GI/G/1/0 loss system. However, one can generalize in another direction without losing the renewal property of the overflow process. Namely, the overflow process from a $GI/M_{(n)}/s/0$ queue, $M_{(n)}$ indicating state-dependent service rates, is renewal as shown by DESCLOUX [11], who also develops procedures for determining the moments of the interoverflow time distribution.

This note is concerned with a variant of Descloux's model. Concretely, we will analyze the overflow process from a single server queueing system with finite waiting room of size $K - 1 (K \geq 1)$, for which the arrival and service rates may depend only on the number of customers in the sysem. These rates will be denoted by λ_n and μ_n $(n=0,1,\ldots,K)$, respectively, while the queueing system itself is referred to as $M_{(n)}/M_{(n)}/1/K-1$. We remark that with appropriate interpretation of the service rates this model encompasses any multiserver delay and loss system with arbitrary (state -dependent) arrival and service rates.

The purpose of this paper is twofold. First, in Section 1, we will show that the overflow process from an $M_{(n)}/M_{(n)}/1/K-1$ system is a renewal process of hyperexponential type and we derive an expression for the Laplace-Stieltjes transform of the interflow time distribution. Then, in Section 2, we will exhibit that this knowledge may advantageously be used to examine Markovian queueing systems where an overflow process from one queue is the arrival process to another. One such system, to wit $(M/M/s/r)_{\text{overflow}}/M/\infty$, will be studied in detail.

2. THE MAIN RESULT

Consider an $M_{(n)}/M_{(n)}/1/K-1$ system $(K \geq 1)$ with arrival rates λ_n and service rates μ_n $(n=0,1,\ldots,K)$, and let $X(t)$ denote the number of customers in the system at time t. Then $\{X(t)\}$ is a birth-death process with state space $S = \{0,1,\ldots,K\}$, birth rates λ_n in state n $(n=0,1,\ldots,K-1)$ and 0 in state K, and death rates μ_n in state n $(n=1,2,\ldots,K)$ and 0 in state 0. Assume that customers overflow at the instants $T_0 = 0$, T_1, T_2, ... and let $U_n = T_n - T_{n-1}$ $(n \geq 1)$ denote the nth interoverflow interval. Regarding the point process $\{T_0, T_1, T_2, \ldots\}$ we have the following result.

THEOREM. *The overflow process from an* $M_{(n)}/M_{(n)}/1/K-1$ *queue* $(K \geq 1)$ *with state-dependent arrival and service rates* λ_n *and* μ_n, *respectively, constitutes a renewal process, the interoverflow distribution* $F(t)$ *being a mixture of* $K+1$ *distinct exponential distributions. The Laplace-Stieltjes transform* $\phi(z)$ *of* $F(t)$ *is given by*

$$(1) \qquad \phi(z) \equiv \int_0^\infty e^{-zy} dF(t) = Q_k(-z)/Q_{K+1}(-z), \quad z \geq 0,$$

where Q_K *and* Q_{K+1} *are polynomials of degree* K *and* $K+1$, *respectively, defined by the recurrence relations*

$$(2) \qquad \begin{aligned} &Q_{-1}(x) = 0, \quad Q_0(x) = 1 \\ &\lambda_n Q_{n+1}(x) = (\lambda_n + \mu_n - x) Q_n(x) - \mu_n Q_{n-1}(x), \quad n = 0,1,\ldots,K. \end{aligned}$$

Finally, the intensity ν *of the overflow process is given by*

$$(3) \qquad \nu \equiv \{\int_0^\infty t\, dF(t)\}^{-1} = \lambda_K \pi_K \{\sum_{n=0}^{K} \pi_n\}^{-1},$$

where

$$(4) \qquad \pi_0 = 1 \text{ and } \pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, \quad n \geq 1.$$

PROOF. Consider a birth-death process $\{X'(t)\}$ with state space $S' = \{0,1,\ldots K,K+1\}$, birth-rates λ_n in state n $(n=0,1,\ldots,K)$ and 0 in state $K+1$, and death rates 0 in state 0, μ_n in state n $(n=1,2,\ldots,K)$ and 0 in state $K+1$, so that $K+1$ is an absorbing state for $\{X'(t)\}$. Suppose that $\{X'(t)\}$ starts at $t = T$ with initial state K and let T_a denote the instant at which absorption into state $K+1$ takes place. Then, evidently, the process $\{X'(t),\ T \leq t < T_a\}$ is a probabilistic replica of the process $\{X(t),\ T_n \leq t < T_{n+1} \mid T_n = T\}$. Since $T_a - T$ is independent of n, it follows that $U_1, U_2, \ldots$ are independent random variables with a common distribution $F(t)$, which equals the distribution of the time until absorption of the process $\{X'(t)\}$ when the initial state is K. So the overflow process is (intrinsically) a renewal process of phase type (cf. NEUTS [31]). The more specific characterization of the theorem is obtained if we interprete $F(t)$ as the first passage time distribution from state K into state K+1 of $\{X'(t)\}$. It then follows from a result of KARLIN and McGREGOR [16, 17] (see also KEILSON [18]) that the Laplace-Stieltjes transform of $F(t)$ is given by (1) and (2). Now writing $R_{-1}(x) = 0$, $R_0(x) = 1$ and

$$(5) \qquad R_{n+1}(x) = (-1)^n \lambda_0 \lambda_1 \cdots \lambda_n Q_{n+1}(x), \quad n = 0,1,\ldots,K,$$

we see that the polynomials $\{R_n(x)\}$ satisfy a three term recurrence formula of the form

$$(6) \qquad R_{n+1}(x) = (x-a_n)R_n(x) - b_n R_{n-1}(x), \quad n \geqq 0,$$

with $b_0 = 0$ and $b_n = \lambda_{n-1}\mu_n > 0$ $(n>0)$, so that they constitute part of an orthogonal system with respect to a positive definite moment functional (see, e.g., CHIHARA [6] Theorem I.4.4). This implies that the zeros of $R_n(x)$ (and hence of $Q_n(x)$) are real and distinct ([6] Theorem I.5.2). Further, since $a_n = \lambda_n + \mu_n$, the parameters a_n and b_n satisfy a criterion due to Stieltjes ([6] p. 47) implying that the zeros of $Q_n(x)$ are positive. A further appeal to the theory of orthogonal polynomials ([6] p. 29) yields that the partial fraction decomposition

$$(7) \qquad \frac{Q_K(-z)}{Q_{K+1}(-z)} = \sum_{n=1}^{K+1} \frac{\omega_n z_n}{z+z_n},$$

where the z_n $(n=1,2,\ldots,K+1)$ are the (positive) zeros of Q_{K+1}, has

$$(8) \qquad \omega_n = -\frac{1}{z_n} \frac{Q_K(z_n)}{Q'_{K+1}(z_n)} > 0.$$

Also, by (7) and the fact that $Q_n(0) = 1$, we have $\Sigma\, \omega_n = 1$, as it should be. So

$$F(t) = \sum_{n=1}^{K+1} \omega_n(1-\exp(-z_n t)),\ t \geq 0, \tag{9}$$

a hyperexponential distribution of order K+1 with distinct parameters for the components.

Finally, since $\nu = -1/\phi'(0)$, we obtain from (1)

$$\nu^{-1} = Q'_{K+1}(0) - Q'_K(0). \tag{10}$$

The result (3) now readily follows from the recurrence relations (2). □

REMARK 1. Substitution of $\lambda_n = \lambda$ and $\mu_n = n\mu$ $(n=0,1,\ldots,K)$ leads to results which are easily seen to coincide with those of PALM [32] and others on the overflow process from an M/M/K/0 loss system.

REMARK 2. Various sources give procedures for and numerical experience with the problem of determining the zeros of the polynomial Q_{K+1}. We mention KUCZURA [24] for the M/M/K/0 case, and MACHIHARA [25] for the general case.

3. AN APPLICATION: $(M/M/s/r)_{overflow}/M/\infty$

We consider an M/M/s/r queue (s servers, r waiting places) with arrival rate λ and service rate μ per server, and let $\alpha = \lambda/\mu$. The overflow process from this queue is offered to an infinite server system also with service rate μ per server, and we are interested in the stationary distribution $\{p(i),\ i=0,1,\ldots\}$ of the number of busy servers in the secondary system.

This model is of importance in a teletraffic context, where it is customary to characterize a stream of calls by the trunk occupancy distribution it induces on an infinite trunk group. For $r = 0$ the model is a classical one (KOSTEN [20], WILKINSON [44]). The general case has been the subject of a paper by RATH and SHENG [36] who describe an approximative procedure for determining the required distribution. Exact analyses of the model have been performed by BASHARIN [1] and HERZOG and KÜHN [13]. They give algorithmic solutions to the problem of determining the moments of the stationary busy-server distribution, the variance of this distribution being explicitly determined by Herzog and Kühn. Both these analyses are based on the equilibrium equations for the joint probabilities p(i,j) of i customers in the M/M/s/r system and j busy servers in the infinite server system. In this section we will show that one can derive explicit expressions for the binomial moments

(11) $$B_k = \sum_{i=k}^{\infty} \binom{i}{k} p(i), \quad k = 1,2,\ldots$$

with relative ease by exploiting the overflow theorem of the previous section and standard results for the GI/M/∞ system. Before elaborating on this approach we remark that it does not seem possible to obtain the explicit results of this section by applying the techniques of RAMASWAMI and NEUTS [35] for the system PH/G/∞. On the other hand, KOSTEN's [21] results for the system MDF/M/∞ can be used to obtain our results but only at the cost of additional efforts.

In concurrence with previous notation we let F(t) denote the interoverflow distribution of the M/M/s/r queue and $\phi(z)$ its Laplace-Stieltjes transform; also, ν^{-1} will denote the mean interoverflow time. The classical results of TAKÁCS [40, 42] and COHEN [7] for the system GI/M/∞ then state that

(12) $$B_k = \frac{\nu}{k\mu} \prod_{j=1}^{k-1} \kappa_j, \quad k = 1,2,\ldots,$$

where the empty product is interpreted as unity and

(13) $$\kappa_j = \phi(j\mu)/(1-\phi(j\mu)), \quad j = 1,2,\ldots .$$

Application of our theorem with $\lambda_n = \lambda$ and $\mu_n = \min(n,s)\mu$ $(n=0,1,\ldots,s+r)$ now yields

(14) $$\nu = \frac{\lambda\alpha^{s+r}}{s!s^r} \left\{ \sum_{n=0}^{s} \frac{\alpha^n}{n!} + \frac{\alpha^s}{s!} \frac{1-(\alpha/s)^r}{1-\alpha/s} \right\}^{-1}$$

and

(15) $$\phi(z) = Q_{s+r}(-z)/Q_{s+r+1}(-z),\ z \geqq 0.$$

According to KARLIN and McGREGOR [15] we have

(16) $$Q_n(\mu x) = c_n(x,\alpha),\ n = 0,1,\ldots,s$$

and

(17) $$Q_{s+n}(\mu x) = (\frac{s}{\alpha})^{n/2}\{Q_s(\mu x)U_n(\frac{\alpha+s-x}{2\sqrt{\alpha s}}) - (\frac{s}{\alpha})^{\frac{1}{2}}Q_{s-1}(\mu x)U_{n-1}(\frac{\alpha+s-x}{2\sqrt{\alpha s}})\},$$
$$n = 0,1,\ldots,r+1.$$

Here the c_n are Charlier polynomials with parameter α, defined by the recurrence relation

(18) $$c_{-1}(x,\alpha) = 0,\ c_0(x,\alpha) = 1$$
$$-xc_n(x,\alpha) = nc_{n-1}(x,\alpha) - (n+\alpha)c_n(x,\alpha) + c_{n+1}(x,\alpha),\ n \geqq 1,$$

and the U_n Chebyshev polynomials of the second kind, recurrently defined by

(19) $$U_{-1}(x) = 0,\ U_0(x) = 1$$
$$2xU_n(x) = U_{n-1}(x) + U_{n+1}(x),\ n \geqq 1$$

(cf. CHIHARA [6]). Writing

(20) $$U_n(\frac{\alpha+s-x}{2\sqrt{\alpha s}}) = (\frac{\alpha}{s})^{n/2}v_n(-x,\alpha),\ n \geqq 0,$$

and subsequently suppressing the parameter α in c_n and v_n, we readily arrive at

$$(21)\qquad \phi(j\mu) = \frac{\alpha c_s(-j)v_r(j) - sc_{s-1}(j)v_{r-1}(j)}{\alpha c_s(-j)v_{r+1}(j) - sc_{s-1}(j)v_r(j)},\ j \geqq 1.$$

Now exploiting another recurrence relation for Charlier polynomials, viz.,

$$(22)\qquad c_n(x+1,\alpha) - c_n(x,\alpha) = -\frac{n}{\alpha}c_{n-1}(x,\alpha),\quad n \geqq 0,$$

(see, e.g., JAGERMAN [14]) for n = s, we obtain the following expression for κ_j.

$$(23)\quad \kappa_j = \frac{(v_r(j)-v_{r-1}(j))c_s(-j)+v_{r-1}(j)c_s(-j+1)}{(v_{r+1}(j)-2v_r(j)+v_{r-1}(j))c_s(-j)+(v_r(j)-v_{r-1}(j))c_s(-j+1)},\ j \geqq 1.$$

For completeness' sake we note that $v_n(j)$ may be written as

$$(24)\qquad v_n(j) = (\gamma_2\gamma_1^{-n} - \gamma_1\gamma_2^{-n})/(\gamma_2 - \gamma_1),$$

where

$$(25)\qquad \gamma_1,\ \gamma_2 = (\alpha+s+j \pm \sqrt{(\alpha+s+j)^2-4\alpha s})/2s.$$

For computational purposes, however, the recurrence relation

$$(26)\qquad \begin{aligned} &v_{-1}(j) = 0,\ v_0(j) = 1 \\ &(\alpha+s+j)v_n(j) = \alpha v_{n+1}(j) + sv_{n-1}(j),\ n \geqq 0, \end{aligned}$$

which follows from (19) and (20), may be more useful. Similarly, an explicit expression for $c_s(-j)$ is given by

$$(27)\qquad c_s(-j) = \sum_{n=0}^{s} \binom{s}{i}\binom{j}{i}\frac{i!}{\alpha^i}$$

(see e.g., [14]), but for numerical work one had better use the recurrence formulas (18) and (22).

Thus (12), (14) and (23) give us expressions for the binomial moments B_k. Specifically, we have for the mean number of busy servers M

(28) $$M = B_1 = \nu/\mu,$$

which follows also directly from Little's formula. Further, noting that

(29) $$c_n(-1,\alpha) = 1/E_n(\alpha), \; n \geqq 0,$$

where

(30) $$E_n(\alpha) = \frac{\alpha^n}{n!}\{1 + \frac{\alpha^1}{1!} + \frac{\alpha^2}{2!} + \ldots + \frac{\alpha^n}{n!}\}^{-1}$$

is the Erlang loss function (see [14]), we obtain for the variance V of the number of busy servers

(31) $$\begin{aligned} V &= 2B_2 + M - M^2 \\ &= \frac{\nu}{\mu}\left(1 - \frac{\nu}{\mu} + \frac{v_r(1)-v_{r-1}(1)+v_{r-1}(1)E_s}{v_{r+1}(1)-2v_r(1)+v_{r-1}(1)+(v_r(1)-v_{r-1}(1))E_s}\right), \end{aligned}$$

where $E_s = E_s(\alpha)$. This expression may be shown to be identical with formula (2.12) of HERZOG and KÜHN [13]. Substitution of r = 0 in (31) immediately yields the Molina-Nyquist result (see WILKINSON [44] or COOPER [9]), while letting r tend to infinity in the expression for V/M (this quantity is called 'peakedness' in teletraffic theory) and using the recurrence formula

(32) $$E_{n+1}(\alpha) = \alpha E_n(\alpha)/(n+\alpha E_n(\alpha)), \; n > 0,$$

for the Erlang loss function (see [14]), leads to Kokotushkin's formula (BASHARIN [1]).

REMARK 1. The quantities κ_j of (13) are also the basic elements in the expressions for the binomial moments of the stationary busy-server distribution in the system GI/M/K/0 (see, e.g., TAKÁCS [42]). Hence, by substitution of (23) in these formulas we can generalize the results of BECH [2] and BROCKMEYER [4] (see also SCHEHRER [38]), who analyze the system $(M/M/s/r)_{\text{overflow}}/M/K/0$ for $r = 0$ on the basis of equilibrium equations.

REMARK 2. A further generalization of the model is obtained when we assume that next to the overflow process an independent Poisson stream of customers arrives at the secondary system. The problem of finding the stationary busy-server distribution of the secondary system may then be tackled by observing that between arrivals from the overflow process the number of busy servers $Y(t)$ behaves as a birth-death process, so that, actually, $\{Y(t)\}$ is a 'Markovian regenerative process' (COHEN [8]) or a 'piecewise Markov process' (KUCZURA [23]), the latter setting being somewhat more general. Since, by our theorem, we have at our disposal the Laplace-Stieltjes transform of the interoverflow distribution, techniques similar to those of KUCZURA [22, 24] may be employed to solve the problem. In this context it is interesting to note that MORRISON ([27]-[30]) studies similar models purely on the basis of equilibrium equations for the combined system of two queues, whose dimensions he substantially reduces. It may be shown, at least when one is interested in the stationary busy-server distribution for the system $M + (M/M/s/0)_{\text{overflow}}/M/K/0$, that Morrison's approach requires approximately the same amount of numerical work as Kuczura's method.

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